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Conformal and Superconformal Mechanics

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ABSTRACT

We investigate the conformal and superconformal properties of a non-relativistic spinning particle propagating in a curved background coupled to a magnetic field and with a scalar potential. We derive the conditions on the couplings for a large class of such systems which are necessary in order their actions admit conformal and superconformal symmetry. We find that some of these conditions can be encoded in the conformal and holomorphic geometry of the background. Several new examples of conformal and superconformal models are also given.

1. Introduction

Conformal transformations leave the metric of a manifold invariant up to a coordinate dependent scale. Many theories exhibit conformal invariance and the classical and quantum consequences of this have been extensively explored in the literature. A celebrated application is the use of superconformal two-dimensional sigma models to describe the consistent propagation of superstrings in curved backgrounds.

More recently, there has been much interest in classical and quantum mechanical systems which exhibit conformal symmetry following some earlier work by de Alfaro, Fubini and Furlan [1]. Superconformal models have been examined in [2-8]. Such systems are sigma models that describe the propagation of a non-relativistic spinning particle in a curved background which is the sigma model target space. Conformal sigma models may have applications in the context of AdS/CFT correspondence adapted for $AdS_2 \times M$ backgrounds [9]. Another application for these models is in the study of the radial motion of non-relativistic test particles with non-vanishing angular momentum near the horizon of extremal Reissner- Nordström black holes [6, 7]. It turns out that the Hamiltonians of the test particles are either given by that of [1] or by a suitable generalization [10]; the target spaces of all these models are flat but they have *scalar* potentials. In addition, supersymmetric [11, 12] and superconformal [13-15] one-dimensional sigma models have found applications in the study of the moduli spaces of black holes in four and five dimensions [16-18, 12, 13-15, 19] that preserve a fraction of spacetime supersymmetry. In particular, it has been shown in [13-15] that for a certain class of black holes the effective theory is described by a supersymmetric sigma model which at small black hole separations it exhibits superconformal symmetry. In this case, the target space of the relevant sigma model is curved and there is no scalar potential. A different application of (two-dimensional) sigma model is in the context of the moduli space of Calabi-Yau black holes [20]. Some other applications of one-dimensional supersymmetric sigma models have been in the light-cone quantization

of supersymmetric theories [21].

It has been known for sometime that there is an interplay between the number of supersymmetries of sigma models and the geometry of their target spaces. The supersymmetric one-dimensional sigma models have been investigated in [11]. It was found that supersymmetry in this case imposes weaker conditions on the geometry on the target spaces than those imposed on the target spaces of sigma models with the same number of supersymmetries but in two or more dimensions. It turns out that in one-dimension there is some flexibility in the derivation of the various conditions that supersymmetry imposes on the couplings of the theory. This makes the analysis of the conditions rather involved and a complete classification of all possibilities has not been done. Some new one-dimensional supersymmetric sigma models have recently been constructed in [22].

Superconformal symmetry imposes further conditions in addition to those of supersymmetry on the target space of one-dimensional sigma models. These additional conditions were first explored in [13, 15] for a class of supersymmetric sigma models which did not have a coupling to a magnetic field or scalar potential. In particular it was found that a necessary condition for a sigma model to have superconformal symmetry is that the sigma model manifold admits a homothetic motion.

In this paper, we investigate the conditions for a non-relativistic spinning particle coupled to a *magnetic field*, with a *scalar potential* and propagating in a curved background to admit conformal and superconformal symmetry. We shall carry out our analysis in the Lagrangian formalism. We first begin with the bosonic spinning particle. The conformal group of the real line is the group of its diffeomorphisms. Such transformations do not leave the non-relativistic spinning particle action invariant unless they are appropriately compensated with diffeomorphisms of the background. We derive the conditions that the background should satisfy in order to admit such diffeomorphisms which induce conformal symmetries in the action of the spinning particle. In particular we find that the necessary conditions for the

spinning particle action to be invariant under the $SL(2,\mathbb{R})$ subgroup of the conformal group are that either the sigma model manifold admits a homothetic motion or it admits two commuting homothetic motions. We apply our results to the case of a charged particle in the presence of a Dirac monopole and to the propagation of a particle in a conical space. We also give the various conserved charges and compute their poisson bracket algebra. We then extend our analysis to the superconformal case. For this we consider a class of supersymmetric non-relativistic spinning particles coupled to a magnetic field and with a scalar potential. To incorporate the latter coupling into the action, we include a fermionic multiplet in analogy with similar results in two-dimensional sigma models [23]. The scalar potential is the length of section of a vector bundle over the background manifold. We first give the conditions for the $N=1$ supersymmetric models to admit a superconformal extension. In particular, we investigate the conditions for the models to be invariant under the $Osp(1|2)$ subgroup of the one-dimensional $N=1$ superconformal group. The conditions for $Osp(1|2)$ invariance are closely related to those of the $SL(2,\mathbb{R})$ invariant bosonic models. We give some examples of $Osp(1|2)$ invariant models. Next we extend our analysis to a class of $N=2B$ and $N=4B$ supersymmetric one-dimensional systems with a magnetic field and scalar potential. In particular we derive the conditions for such models to be invariant under the $SU(1,1|1)$ and $D(2,1|\alpha)$ subgroups of the $N=2B$ and $N=4B$ superconformal groups in one-dimension, respectively. In the special case of sigma models without fermionic superfields and without a coupling to a magnetic field the above conditions are related to those in [13].

This paper has been organized as follows: In section two, we derive the conditions for conformal invariance of a bosonic spinning particle. In section three, we give the conserved charges and compute their poisson bracket algebra. In section four, we investigate the conditions for a $N=1$ supersymmetric spinning particle to admit superconformal symmetry. In section five, we give the conditions for a $N=2B$ supersymmetric spinning particle to admit superconformal symmetry. In section six, we investigate the conditions for a $N=4B$ supersymmetric spinning

particle to admit superconformal symmetry and in section seven we give our conclusions. In appendix A, we present a geometric interpretation for the conformal transformations that we are considering. In appendix B, we give the most general action for a $N = 1$ supersymmetric sigma model with couplings which are at most quadratic in the velocities of the bosonic fields.

2. Conformal Bosonic Sigma Models

2.1. Conformal Invariance

The conformal group, $\mathrm{Conf}(\mathbb{R}^n)$ of n-dimensional Euclidean spaces, $\mathbb{R}^n$, for $n \geq 3$, is a finite dimensional Lie group which is isomorphic to $O(n+1,1)$. But in one- and two-dimensional Euclidean spaces, the conformal group is infinite dimensional. In one-dimension the conformal group $\mathrm{Conf}(\mathbb{R})$ is isomorphic to the diffeomorphism group, $\mathrm{Diff}(\mathbb{R})$ of the real line. So the Lie algebra $\mathrm{conf}(\mathbb{R})$ of $\mathrm{Conf}(\mathbb{R})$ is the set of vector fields on $\mathbb{R}$ equipped with the Lie bracket. The set of vector fields on $\mathbb{R}$ can be identified with the set of real functions on $\mathbb{R}$, and so $\mathrm{conf}(\mathbb{R}) = C^\infty(\mathbb{R})$. In this identification a vector field on $\mathbb{R}$ is associated to its component in $C^\infty(\mathbb{R})$. So, it is not realistic to expect that we shall be able to construct models which are invariant under the full conformal group in one-dimension; invariant models under the full conformal group will be necessarily diffeomorhic invariant and so a coupling to gravity may have to be included. Therefore of interest are the models which are invariant under a proper subgroup of the conformal group in one-dimension. One such case is the subalgebra of $\mathrm{conf}(\mathbb{R})$ generated by the polynomials $P[t]$ on the real line. This case includes the dilations L and the special conformal transformations T which have been examined in the context of one-dimensional sigma models in [13]. The translation H along the worldline together with L and T generate the $sl(2,\mathbb{R})$ subalgebra,

$$[L,T] = T, \qquad [H,L] = H, \qquad [H,T] = \frac{1}{2}L \ , \tag{2.1}$$

of the conformal algebra in one-dimension. The systems that are invariant under

the subalgebra (2.1) of the conformal algebra of the real line were called conformal or conformally invariant. However in what follows we shall not make this restriction. We shall investigate conformal invariance under the full conformal group of the real line or a subgroup of it that contains (2.1). As we shall see though many interesting systems are invariant only under the action of the subalgebra (2.1).

Our approach to investigate the conformal symmetries of the spinning particle is geometric, so it is more convenient to use the sigma model language to describe the various computations. Let M be a Riemannian manifold equipped with a metric g, a locally defined (up to an exact one-form) one-form A and a locally defined (up to a constant) function V. The Lagrangian of a one-dimensional sigma model with target space M is

$$L = \frac{1}{2} g_{ij} \frac{d}{dt} q^i \frac{d}{dt} q^j + A_i \frac{d}{dt} q^i - V(q) \ , \tag{2.2}$$

where t is a coordinate on the worldline $\mathbb{R}$ and q are the sigma model maps from $\mathbb{R}$ into M. The couplings A and V are those of a magnetic field and a scalar potential, respectively. Such Lagrangian describes a non-relativistic spinning particle coupled to a magnetic field A with scalar potential V and propagating in the background with metric g. The field equations are

$$\nabla_t \frac{d}{dt} q^i - g^{ik} F_{kj} \frac{d}{dt} q^j + g^{ij} \partial_j V = 0 \ , \tag{2.3}$$

where $F_{ij} = 2\partial_{[i} A_{j]}$. Note that the field equations are expressed in terms of globally defined tensors on M.

The action of the sigma model (2.2) is invariant under worldline translations $\delta t = \epsilon$ which are generated by $a = d/dt$. The rest of the conformal transformations do *not* leave (2.2) invariant. However we can circumvent this by introducing a transformation for q which is generated by a vector field X in M. We also allow X to depend explicitly on t. The vector fields X generate an one-parameter family of diffeomorphism of M and generic diffeomorphisms do *not* leave the sigma model

action invariant, i.e. there are not conserved charges associated with them. But if a conformal transformation is appropriately compensated with a diffeomorphism of M, then under certain conditions their combination can leave the action invariant. So we seek the conditions for the Lagrangian (2.2) to be invariant up to possibly surface terms under the transformations[⋆]

$$\begin{aligned} \bar{\delta}_\epsilon q^i &= \epsilon X^i(t,q) \\ \bar{\delta}_\epsilon t &= \epsilon a(t) \ , \end{aligned} \tag{2.4}$$

where ϵ is a constant infinitesimal parameter. The induced transformation on q is

$$\delta_\epsilon q^i = -\epsilon a(t) \frac{d}{dt} q^i + \epsilon X^i(t,q) \ . \tag{2.5}$$

To continue we note that if $f = f(t, q(t))$, then the total derivative $\frac{d}{dt}$ of f with respect to t is

$$\frac{d}{dt} f = \partial_t f + \partial_i f \frac{d}{dt} q^i \ , \tag{2.6}$$

where ∂_t denotes differentiation which acts only on the explicit dependence of f on t. For functions that depend *only* on t, like q and a, $\frac{d}{dt}$ is the same as ∂_t, and in what follows we shall use either one or the other notation at convenience. The commutator of two such transformations yields

$$[\delta_\epsilon, \delta_{\epsilon'}] q^i = -\epsilon\epsilon' [a, a']^t \frac{d}{dt} q^i + \epsilon\epsilon' \big(a \partial_t X'^i - a' \partial_t X^i + [X, X']^i \big) \ , \tag{2.7}$$

where $[a, a']$ and $[X, X']$ are the Lie brackets of the vector fields a, a' and X, X', respectively. The commutator is a new transformation on q induced by the Lie bracket of the vector fields a, a' as expected and by a new vector field Y on M with components

$$Y^i = a \partial_t X'^i - a' \partial_t X^i + [X, X']^i \ . \tag{2.8}$$

Observe that this new vector field is of the same type as that of X and X' and so the algebra of transformations closes. The Lagrangian (2.2) is invariant under the

⋆ The geometric interpretation of these transformations is given in appendix A.

transformations (2.5) provided that

$$\begin{aligned} \nabla_{(i}X_{j)} &= \frac{1}{2}\partial_t a g_{ij} \\ \partial_t X^j g_{ji} + X^j F_{ji} &= \partial_i f \\ \partial_t a V + X^i \partial_i V &= -\partial_t f \ , \end{aligned} \tag{2.9}$$

where $f = f(t, q)$ is a function on $\mathbb{R} \times M$. The function f arises because the Lagrangian (2.2) should be invariant up to a surface term. In general f could be chosen to depend on the time derivatives of q as well, however, since our Lagrangian is at most quadratic in the time derivatives of the fields the above choice of f suffices.

2.2. Special Cases

Next we consider the class of conformal transformations generated by the polynomials $P[t]$ on $\mathbb{R}$. We also assume that the vector fields X and the boundary term f are polynomials in t. So we have

$$\begin{aligned} a(t) &= \sum_{n=0}^{I} a_n t^n \\ X(t, q) &= \sum_{n=0}^{J} X_n(q) t^n \quad . \\ f(t, q) &= \sum_{n=0}^{K} f_n(q) t^n \end{aligned} \tag{2.10}$$

Substituting these into the equations for the invariance of the action we find that

$$J \geq I + 1 \tag{2.11}$$

and

$$\begin{aligned} \nabla_{(i}X_{nj)} &= \frac{1}{2}(n+1)a_{n+1}g_{ij} \\ (n+1)X^j_{n+1}g_{ji} + X^j_n F_{ji} &= \partial_i f_n \\ (n+1)a_{n+1}V + X^i_n \partial_i V &= -(n+1)f_{n+1} \ , \end{aligned} \tag{2.12}$$

For example, if we take

$$\begin{aligned} a &= t \\ X(t,q) &= Z(q) \end{aligned} \tag{2.13}$$

which are the dilations in $\mathbb{R}$, we find that

$$\begin{aligned} \nabla_{(i}Z_{j)} &= \frac{1}{2}g_{ij} \\ Z^j F_{ji} &= \partial_i f(q) \\ V + Z^i\partial_i V &= 0 \ , \end{aligned} \tag{2.14}$$

up to a possibly constant shift of V. So Z generates a homothetic motion on M which leaves F invariant. For special conformal transformations

$$a = t^2 \ . \tag{2.15}$$

A choice for X is

$$X(t,q) = tY(q) \ . \tag{2.16}$$

Substituting these into (2.12), we find

$$\begin{aligned} \nabla_{(i}Y_{j)} &= g_{ij} \\ Y^j g_{ji} &= \partial_i f_0 \\ Y^j F_{ji} &= \partial_i f_1 \\ 2V + Y^i\partial_i V &= -f_1 \ , \end{aligned} \tag{2.17}$$

So Y is a homothetic vector field on M which leaves F invariant and is generated by a 'homothetic potential' f_0. We can choose the vector field Y above to be different from Z in (2.14). But a minimal choice is to set

$$Y = 2Z \ . \tag{2.18}$$

In such which case, $f_1 = 0$ and we find that

$$
\begin{aligned}
\nabla_{(i}Z_{j)} &= \frac{1}{2}g_{ij} \\
Z^j g_{ji} &= \frac{1}{2}\partial_i f_0 \\
Z^j F_{ji} &= 0 \\
V + Z^i\partial_i V &= 0 \ .
\end{aligned}
\tag{2.19}
$$

These are the conditions for a sigma model to be invariant under the dilations and special conformal transformations generated by a single dilatation Z in the target space M. In particular, observe that

$$
Z^j g_{ji} = \frac{1}{2}\partial_i f_0 \tag{2.20}
$$

where f_0 is a homothetic potential. No further conditions are required by the closure of the algebra of transformations. For $F = 0$, this reproduces the results of [13]. If $M = \mathbb{R}^d$ and $F = 0$, then $X = \frac{1}{2}q^i\partial_i$ and V scales with degree -2. For $d = 1$, this gives $V = q^{-2}$ which is the standard potential of conformal mechanics [1].

Next let us consider the invariance of the action under the transformation generated by

$$
\begin{aligned}
a &= t^{n+1} \\
X &= (n+1)t^n Z
\end{aligned}
\tag{2.21}
$$

for $n \geq 2$, where Z generates a homothetic motion. Clearly the first condition in (2.12) is satisfied. The remaining conditions are

$$
\begin{aligned}
(n+1)nZ^j g_{ji} &= \partial_i f_{n-1} \\
(n+1)Z^J F_{ji} &= \partial_i f_n \\
\partial_i f_k &= 0, \qquad k \neq n, n-1 \\
V + Z^j\partial_j V &= -f_{n+1} \\
f_k &= 0, \qquad k \neq n+1
\end{aligned}
\tag{2.22}
$$

Now since $V + Z^j \partial_j V = 0$, (2.22) implies that $f = 0$. So we conclude that

$$Z^j g_{ji} = 0 \tag{2.23}$$

and if the metric is non-degenerate, then $Z = 0$. So there are no additional such symmetries generated by a single homothetic motion in M apart from those of dilations and special conformal transformations above.

Apart from the minimal case we have considered above, we can also choose Z and Y in (2.14) and in (2.17), respectively, to be linearly independent. Both Z and Y generate homothetic motions on M. It is straightforward to observe from (2.14) and (2.17) that although Y is associated to a homothetic potential, Z does not. The algebra of the transformations generated by Z and Y closes without the addition of other transformations provided that

$$[Z, Y] = 0 \ . \tag{2.24}$$

The algebra of transformations is given by (2.1). One can easily extend the above results to the case that the sigma model manifold admits a homothetic group action which generates more than two vector fields on M.

To summarize, the necessary conditions for a spinning particle propagating in a curved background to admit a $sl(2, \mathbb{R})$ are either

the existence of a homothetic motion in the background generated by a homothetic potential or the presence of two commuting homothetic motions. Additional conditions though are satisfied by the other couplings of the theory like that of the magnetic field and scalar potential.

2.3. Examples and Applications

Dirac Monopole

As an example, we consider a non-relativistic particle in $\mathbb{R}^3$ coupled to a Dirac monopole located at the origin. We take $Z = \frac{1}{2}q^i\partial_i$. Clearly the first condition in (2.19) is satisfied ($g_{ij} = \delta_{ij}$). Choosing $f_0 = \frac{1}{2}|q|^2$, we can easily verify the second condition in (2.19). Moreover since

$$F_{ij} = -\frac{\epsilon_{ijk}q^k}{|q|^3} \tag{2.25}$$

we find that $Z^j F_{ji} = 0$ and the third condition in (2.19) is also satisfied. Finally the potential V can either vanish or be any homogeneous function of q of degree -2. Therefore the system is invariant under dilatations and the special conformal transformations on $\mathbb{R}$ which together with the translations in $\mathbb{R}$ generate an $sl(2,\mathbb{R})$ symmetry. Apart from these symmetries the action is also invariant under $so(3)$ rotations provided the potential either vanishes or $V = \frac{1}{|q|^2}$. These are generated by $X_i = \epsilon_{ijk}q^j\partial_k$ and $a = 0$. One can easily show that $[Z, X_i] = 0$. As a result the transformations generated by the rotations commute with those of $sl(2,\mathbb{R})$. The full symmetry of the system is $so(3) \oplus sl(2,\mathbb{R})$

Particles propagating in Cones

A large class of manifolds M that admit a homothetic motion are those that are cones over another manifold N, i.e. $M = C(N)$ and the metric of M can be written as

$$ds^2(M) = dr^2 + r^2 ds^2(N) \ , \tag{2.26}$$

where r is a radial coordinate. This example has also been considered in [13]. The homothetic vector field is

$$Z = r\frac{\partial}{\partial r} \ . \tag{2.27}$$

Conical spacetimes have found applications in the context of M- and string theories. It is well known that there are M- and string brane solutions for which

near the horizon are $AdS_d \times N$ and at infinity are $\mathbb{R}^{(1,d-2)} \times C(N)$. The manifold N is usually chosen to be a coset space G/H which occurs in various compactifications of eleven- and ten-dimensional supergravity theories.

Next consider a line bundle over N with connection A which we can use as a magnetic field. It is straightforward to observe that all the conditions required for conformal invariance of the associated sigma model are met provided that we choose

$$V = r^{-2} U \ , \tag{2.28}$$

where U is a function on N.

Next if $N = G/H$ and we require invariance of the spinning particle action under G, the metric, magnetic field and scalar potential should be invariant under G. In particular this implies that the function U in the potential V should be constant. The symmetries generated by G and those of $Sl(2,\mathbb{R})$ commute. The symmetry of the system is $SL(2,\mathbb{R}) \times G$.

3. Conserved Charges

3.1. Poisson Bracket Algebra

The conformal transformations investigated in the previous section leave the sigma model action invariant up to surface terms. Taking these surface term into consideration, we find that the conserved charges associated with the (2.5) transformations are

$$C(a,X) = -\frac{1}{2} a g_{ij} \partial_t q^i \partial_t q^j + g_{ij} \partial_t q^i X^j - aV(q) - f \ . \tag{3.1}$$

It is straightforward to show after some computation that they are conserved subject to field equations using (2.9). These charges are well-defined provided that V and f are functions on M and $\mathbb{R} \times M$, respectively. The above charges are not

uniquely defined but they can shift up to a constant. This is because f is defined up to a constant.

The algebra of charges is usually computed by turning into Hamiltonian formalism. However, this need not be the case because a Poisson-like bracket can be defined within the Lagrangian approach[⋆] as

$$\{C(a,X),C(a',X')\} \equiv \delta_\epsilon C(a',X')|_{\epsilon=1} - \delta_{\epsilon'} C(a,X)|_{\epsilon'=1} \ . \tag{3.2}$$

For our conserved charges, we find that

$$\begin{aligned}\{C(a,X),C(a',X')\} = 2\big[&- \frac{1}{2}[a,a']g_{ij}\partial_t q^i \partial_t q^j \\ &+\big(\partial_t X'^j - a'\partial_t X^j + [X,X']^j\big) g_{ji}\partial_t q^i - [a,a']V(q) \\ &-(\partial_t f' a - \partial_t f a' + F_{ij}X^i X'^j + X^j \partial_j f' - X'^j \partial_j f)\big] \\ &\quad - a g_{ij}\mathcal{S}^i X'^j + a' g_{ij}\mathcal{S}^i X^j \ ,\end{aligned} \tag{3.3}$$

where $\mathcal{S}$ are the equations of motion. Neglecting the terms involving the field equations, this algebra can be rewritten as

$$\{C(a,X),C(a',X')\} = 2C([a,a'],[X,X']) + \kappa\big((a,X),(a',X')\big) \ , \tag{3.4}$$

where

$$\kappa\big((a,X),(a',X')\big) = f([a,a'],[X,X']) - (\partial_t f' a - \partial_t f a' + F_{ij}X^i X'^j + X^j \partial_j f' - X'^j \partial_j f) \tag{3.5}$$

is a constant and $f([a,a'],[X,X'])$ is the surface term associated with the commutator of the symmetries generated by (a,X) and (a',X'). If κ does not vanish, then the algebra of charges may develop a central extension. This extension occurs if it cannot be removed by shifting the charges up to constants. The central extension defines an element in the second Lie-algebra cohomology of the symmetry group and vanishes whenever the class is trivial.

⋆ This way of defining Poisson brackets is also known as the covariant canonical approach to classical mechanics.

3.2. Examples

In the special case for which $a = t^{n+1}$ and $X = (n+1)t^n Z$, $n = -1$ (translations), $n = 0$ (dilations) and $n = 1$ (special conformal transformations) investigated in section (2.2), the conserved charges are

$$C_{n+1} = -\frac{1}{2}t^{n+1} g_{ij}\partial_t q^i \partial_t q^j + (n+1)t^n g_{ij}\partial_t q^i Z^j - t^{n+1}V(q) - f \ . \tag{3.6}$$

In particular, we find that

$$\begin{aligned} C_0 &= -\frac{1}{2} g_{ij}\partial_t q^i \partial_t q^j - V(q) \\ C_1 &= -\frac{1}{2} t g_{ij}\partial_t q^i \partial_t q^j + g_{ij}\partial_t q^i Z^j - tV(q) \\ C_2 &= -\frac{1}{2} t^2 g_{ij}\partial_t q^i \partial_t q^j + 2t g_{ij}\partial_t q^i Z^j - t^2 V(q) - f_0 \ , \end{aligned} \tag{3.7}$$

where f_0 is given in (2.19). The algebra of the conserved charges does not develop a central extension because $sl(2, \mathbb{R})$ is semisimple. The conserved charges of a particle in $\mathbb{R}^3$ coupled to a Dirac monopole are

$$\begin{aligned} C_0 &= -\frac{1}{2}\delta_{ij}\partial_t q^i \partial_t q^j - V(q) \\ C_1 &= -\frac{1}{2} t\delta_{ij}\partial_t q^i \partial_t q^j + \delta_{ij}\partial_t q^i Z^j - tV(q) \\ C_2 &= -\frac{1}{2} t^2 \delta_{ij}\partial_t q^i \partial_t q^j + 2t\delta_{ij}\partial_t q^i Z^j - t^2 V(q) - \frac{1}{2}|q|^2 \ . \end{aligned} \tag{3.8}$$

The conserved charges of the rotational symmetries generated by $X_{(i)} = \epsilon_{ijk} q^j \partial_k$ are

$$C_i = \epsilon_{ijk} q^j \partial_t q^k + q_i |q|^{-1} \ . \tag{3.9}$$

The second term in the right-hand-side of (3.9) is present because

$$X^j_{(i)} F_{jk} = -\partial_k (q_i |q|^{-1}) \ . \tag{3.10}$$

The algebra of the charges (3.8) and (3.10) does not develop a central extension because $so(3) \oplus sl(2, \mathbb{R})$ is semi-simple.

It is straight forward to give the charges of the conformal symmetries of a spinning particle propagating in a conical space. For conical spaces over $N = G/H$, the algebra of charges of the system is $sl(2,\mathbb{R})\oplus\mathrm{Lie}G$ provided that G is semisimple. If G is not semisimple and the charges associated with G-invariance are well defined, there is the possibility of a central extension in the algebra of charges of associated with G symmetries. This is due to the presence of the magnetic coupling. For more details see for example [24].

4. N=1 Superconformal Sigma Models

4.1. Superconformal Transformations

In analogy with the bosonic models, we may identify the $N = 1$ superconformal algebra with the superdiffeomorphisms of the $N = 1$ superspace Ξ. Let (t, θ) be the commuting and anti-commuting coordinates of Ξ, respectively. We can construct the even and odd vector fields on Ξ as

$$\begin{aligned} S_e &= a(t)\partial_t + b(t)\theta\partial_\theta \\ S_o &= c(t)\partial_\theta + e(t)\theta\partial_t \ , \end{aligned} \tag{4.1}$$

respectively, where a, b, c, e are functions of t.

As we have already mentioned in the introduction, in many applications one considers the subalgebra $Osp(1|2)$ of vector fields on Ξ with non-vanishing (anti-)commutators

$$\begin{aligned} &\{Q,Q\} = 2H, \qquad [Q,L] = Q, \qquad \{S,Q\} = L, \qquad \{S,S\} = 2T \\ &[L,T] = T, \qquad [L,S] = 2S, \qquad [Q,T] = \frac{1}{2}S, \qquad [S,H] = -\frac{1}{2}Q \\ &[H,L] = H, \qquad [H,T] = \frac{1}{2}L, \qquad [L,T] = T \end{aligned} \tag{4.2}$$

where H is the generator of translation, L is the generator of dilation, T is the generator of special conformal, Q is the generator of supersymmetry and S is the

generator of special superconformal transformations. In terms of vector fields on Ξ this subalgebra is realized[⋆] as follows:

$$
\begin{aligned}
&H=\partial_t, \qquad Q=\partial_\theta+\theta\partial_t \ , \\
&L=t\partial_t+\frac{1}{2}\theta\partial_\theta, \qquad S=\frac{t}{2}Q, \qquad T=\frac{t^2}{4}\partial_t+\frac{t}{4}\theta\partial_\theta \ .
\end{aligned}
\tag{4.3}
$$

Next we introduce the $N=1$ superfields q as functions from the superspace Ξ into the sigma model manifold M. The transformations induced on superfields q by the odd vector fields are

$$
\bar{\delta}q^i=-\alpha\big[c(t)\frac{\partial}{\partial\theta}q+e(t)\theta\frac{\partial}{\partial t}\big]q^i \ , \tag{4.4}
$$

where α is an anti-commuting infinitesimal parameter. Let $D=\frac{\partial}{\partial\theta}-\theta\partial_t$ be the supersymmetry derivative, i.e.

$$
QD+DQ=0, \qquad D^2=-\partial_t \ . \tag{4.5}
$$

In components, the above transformations read

$$
\begin{aligned}
\bar{\delta}q^i&=-\eta c(t)\lambda^i \\
\bar{\delta}\lambda^i&=+\eta e(t)\partial_t q^i \ ,
\end{aligned}
\tag{4.6}
$$

where

$$
\begin{aligned}
q^i&=q^i| \\
\lambda^i&=Dq^i|
\end{aligned}
\tag{4.7}
$$

and the vertical line denotes evaluation at $\theta=0$. Note also that we use the same symbol to denote the superfield and its first component.

⋆ We take the even and odd coordinates of Ξ to be real. Moreover we define the complex conjugation of fermion bi-linears as $(\lambda_1\lambda_2)^*=\lambda_1^*\lambda_2^*$. In this notation bi-linears of real fermions are real without the use of the imaginary unit; compare with the conventional notation of e.g.[11].

The transformations (4.6) do not leave the sigma model action invariant. So as in the bosonic case we introduce compensating target space transformations and then derive the conditions for the combine transformation to leave the sigma model action invariant. In addition, we *require* that the *commutator* of two such transformations on q reproduces the conformal transformations that we have investigated in the bosonic case up to terms involving the fermions λ. We take as a combined transformation

$$\begin{aligned}\delta q^i &= -\alpha c(t)\lambda^i + \alpha X(t,q)^i{}_j\lambda^j \\ \delta\lambda^i &= \alpha e(t)\partial_t q^i + \alpha X^i(t,q)\ ,\end{aligned} \tag{4.8}$$

where α is the anti-commuting infinitesimal parameter, X^i is a vector field in M and $X^i{}_j$ is a tangent space rotation. However it turns out that this is a more general class than it is required. A direct computation of the commutator of two such transformations on q reveals that is a transformation of the type (2.5) provided that we set $X^i{}_j = 0$. So the transformations that we shall consider are

$$\begin{aligned}\delta q^i &= -\alpha c(t)\lambda^i \\ \delta\lambda^i &= \alpha e(t)\partial_t q^i + \alpha X^i(t,q)\ .\end{aligned} \tag{4.9}$$

The commutator of two such transformations is

$$\begin{aligned}[\delta_\alpha, \delta'_\alpha]q^i &= \alpha\alpha'[(c'e + ce')\partial_t q^i + c'X^i + cX'^i] \\ [\delta_\alpha, \delta'_\alpha]\lambda^i &= \alpha\alpha'[(c'e + ce')\partial_t\lambda^i \\ &\quad + (e'\partial_t c + e\partial_t c')\lambda^i + (c'\partial_j X^i + c\partial_j X'^i)\lambda^j]\ .\end{aligned} \tag{4.10}$$

Clearly, the transformation on q induced by the commutator is of the type that we have considered in the bosonic case.

4.2. Invariance of the Action

It remains to investigate the invariance of the action of the $N=1$ supersymmetric sigma models under the transformations (4.9). Before, we investigate this we introduce in addition to the superfield q another fermionic N=1 superfield* ψ. The superfield ψ can be thought of as a section of the vector bundle $S\otimes q^*E$ over the superspace Ξ, where S is the spin bundle over Ξ and E is a vector bundle over M equipped with a fibre metric h and a compatible† connection ∇, i.e. $\nabla h=0$. The introduction of ψ is necessary for the addition of a scalar potential in $N=1$ sigma models. The components of the superfield ψ are a fermion ψ and auxiliary field F, i.e.

$$\psi^a=\psi^a| \qquad F^a=\nabla\psi^a| \tag{4.11}$$

where

$$\nabla\psi^a=D\psi^a+Dq^iB_i{}^a{}_b\psi^b \tag{4.12}$$

and B is a connection of E.

The action for the superfields q and ψ that we shall consider is

$$S=-\int dtd\theta\big[\frac{1}{2}g_{ij}Dq^i\partial_tq^j+\frac{1}{3!}c_{ijk}Dq^iDq^jDq^k+A_iDq^i-\frac{1}{2}h_{ab}\psi^a\nabla\psi^b+ms_a\psi^a\big]\ , \tag{4.13}$$

where g and A are the metric and one-form coupling as the bosonic case, respectively, c_{ijk} is a three-form on M and s_a is a section of the dual of the vector bundle E. The parameter m has the dimension of mass. This is not the most general action for q and ψ (see [11] and appendix B). However, this choice will suffice for our purpose.

$\star$ The superfield ψ has been introduced in the context of two-dimensional sigma models in [25] and in the context of one-dimensional sigma models in [11].

$\dagger$ If another non-compatible connection is chosen, we can always redefine it and rewrite the action in terms of a compatible one, see [26].

To compare the action that we are proposing above to that of the bosonic theory, we expand (4.13) in components and eliminate the auxiliary field using the field equations. Then we find

$$\begin{aligned} S = \int dt \big[& \frac{1}{2} g_{ij} \partial_t q^i \partial_t q^j + \frac{1}{2} g_{ij} \lambda^i \nabla_t^{(+)} \lambda^j - \frac{1}{3!} \partial_{[i} c_{jkm]} \lambda^i \lambda^j \lambda^k \lambda^m \\ & + A_i \partial_t q^i - \frac{1}{2} F_{ij} \lambda^i \lambda^j + h_{ab} \psi^a \nabla_t \psi^b - \frac{1}{4} G_{ijab} \lambda^i \lambda^j \psi^a \psi^b \\ & - m \nabla_i s_a \lambda^i \psi^a - \frac{m^2}{2} h^{ab} s_a s_b \big] \ , \end{aligned} \tag{4.14}$$

where

$$G_{ij}{}^a{}_b = \partial_i B^a_{j\,b} - \partial_j B^a_{i\,b} + B^a_{i\,c} B^c_{j\,b} - B^a_{j\,c} B^c_{i\,b} \tag{4.15}$$

is the curvature of B,

$$\nabla_t^{(+)} \lambda^i = \nabla_t \lambda^i - c^i{}_{jk} \partial_t q^j \lambda^k \ ; \tag{4.16}$$

∇ is associated to the Levi-Civita connection of the metric g. The bosonic part of this action is that of the models considered in section two with scalar potential

$$V(q) = \frac{m^2}{2} h^{ab} s_a s_b \tag{4.17}$$

which is proportional to the length of the section s.

To continue it is convenient to rewrite the transformations (4.9) in terms of superfields as

$$\delta_\eta q^i = -D\eta c(t) Q q^i - \eta (e - c) \partial_t q^i + \eta X^i(t, q) \ , \tag{4.18}$$

where $\eta = \eta(\theta)$ is a commuting constant infinitesimal parameter with components

$$0 = \eta|, \qquad \alpha = D\eta| \ . \tag{4.19}$$

We also take that the transformation induced by the odd vector fields of Ξ on the ψ multiplet together with a compensating transformation generated by a fibre

rotation $L^a{}_b(q,t)$ in E as

$$\delta\psi^a = -D\eta\ell(t)Q\psi^a + \eta L^a{}_b(q,t)\psi^b \ - B_i{}^a{}_b\eta X^i(t,q), \tag{4.20}$$

where $\ell(t)$ is a new function of t which can be chosen to be independent from e and c in (4.18). The conditions for the invariance of the action involving only the superfield q under (4.18) are

$$\begin{aligned}
\nabla_{(i}X_{j)} &= -\partial_t\, c\, g_{ij} \\
\nabla_{[i}X_{j]} &= 0 \\
e &= c \\
2\partial_t X^i + X^i F_{ij} &= 0 \\
X^i c_{ijk} &= 0 \\
\mathcal{L}_X c_{ijk} &= 4\partial_t c\, c_{ijk}\ ,
\end{aligned} \tag{4.21}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to X. Next the conditions for the invariance of the part of the action involving the superfield ψ are

$$\begin{aligned}
X^i(q,t)G_{ij}{}^a{}_b &= 0 \\
L_{ab} &= 0 \\
X^i\nabla_i s_a &= \partial_t c s_a \\
\ell &= c\ .
\end{aligned} \tag{4.22}$$

The commutator of two N=1 superconformal transformations of the components ψ and F of the superfield ψ gives

$$\begin{aligned}
[\delta_\alpha,\delta_\beta]\psi^a &= 2\alpha\beta cc'\nabla_t\psi^a - B_i{}^a{}_b[\delta_\alpha,\delta_\beta]q^i\psi^b \\
[\delta_\alpha,\delta_\beta]F^a &= 2\alpha\beta cc'\nabla_t F^a - \partial_t(cc')\alpha\beta F^a - B_i{}^a{}_b[\delta_\alpha,\delta_\beta]q^i F^b\ .
\end{aligned} \tag{4.23}$$

4.3. Special Cases

A choice of c and X that leads to invariance of the sigma model action under the superconformal algebra (4.2) is

$$\begin{aligned} c &= c_0 + tc_1 \\ X_i(q,t) &= X_i(q) = \partial_i f(q) \ , \end{aligned} \tag{4.24}$$

i.e. X is chosen to be independent from t. Substituting (4.24) into (4.21), the remaining conditions become

$$\begin{aligned} \nabla_{(i}X_{j)} &= -c_1\, g_{ij} \\ e &= c = c_0 + tc_1 \\ X^i F_{ij} &= 0 \\ X^i c_{ijk} &= 0 \\ \mathcal{L}_X c_{ijk} &= 4c_1\, c_{ijk} \ . \end{aligned} \tag{4.25}$$

Similarly substituting (4.24) in (4.22), we find

$$\begin{aligned} X^i G_{ij}{}^a{}_b &= 0 \\ L_{ab} &= 0 \\ X^i \nabla_i s_a &= c_1 s_a \\ \ell &= c_0 + tc_1 \ . \end{aligned} \tag{4.26}$$

The conditions (4.25) and (4.26) are those that are necessary and sufficient for the supersymmetric sigma models to be invariant under the transformations generated by the $Osp(1|2)$ algebra in (4.2). In particular, these conditions imply that X generates a homothetic motion on the target space M, X is expressed in terms of a homothetic potential and in addition it has vanishing contractions with F and G. The first three conditions are precisely the conditions that we have derived in the bosonic case. It remains to derive the condition for the scalar potential V. For

this, we apply the Lie derivative on V with respect to X and use (4.26). We find that

$$X^i \partial_i V - 2c_1 V = 0 \tag{4.27}$$

in agreement with the result in the bosonic case after an appropriate rescaling of X. In the subclass of models without the fermionic multiplet ψ and without a coupling to a magnetic field the above conditions for $Osp(1|2)$ invariance reduce to those of [13].

4.4. Examples

Dirac Monopole

One example of a model that admits $N = 1$ superconformal symmetry is that of a particle in $\mathbb{R}^3$ in the background of a Dirac monopole without scalar potential. It is straightforward to verify that in this case all the conditions are met. Moreover a potential can be added provided that it can be written as the length of a section. To illustrate this, let us consider the case of the model that it is also invariant under $so(3)$ rotations. We introduce a single fermionic multiplet ψ and choose $h_{11} = 1$. We also take $B = 0$. Since the potential in this case is $V = |q|^{-2}$, we can take

$$s = |q|^{-1} \ . \tag{4.28}$$

Then all the conditions are satisfied and such a model is invariant under the action of the $Osp(1|2) \times SO(3)$ group.

Particles propagating in cones

Another example can be found by considering spinning particles propagating in cones, $M = C(N)$, as in the bosonic case in section (2.3). To demonstrate this we first choose the metric in M, Maxwell field F and the vector field that generates the homothetic motion as in the bosonic case. Moreover, we assume that the three-form c vanishes. It remains to construct the scalar potential. For this, we consider

a vector bundle $\tilde{E}$ on N with connection $\tilde{B}$ and fibre metric $\tilde{h}$. Then we pull back E on M using the projection, $\pi \quad M = C(N) \to N$, such that

$$\pi : \quad (r, x) \to x \ , \tag{4.29}$$

where $x \in N$. This projection is well defined everywhere apart from the apex of the cone at $r = 0$. For the geometric data necessary to describe the couplings of the ψ multiplet, we choose $E = \pi^* \tilde{E}$, $B = \pi^* \tilde{B}$ and $h = \pi^* \tilde{h}$. It remains to pick a section in the dual of E. For this we take

$$s = r^{-1} \pi^* \tilde{s} \ , \tag{4.30}$$

where $\tilde{s}$ is a section in the dual vector bundle of $\tilde{E}$. It is straightforward to observe that a sigma model with the above couplings satisfies all the conditions to admit $Osp(1|2)$ superconformal symmetry. If in addition $N = G/H$ and the various other geometric data like the sigma model metric g, the connection B, the fibre metric h and the section s are invariant under the action of G, the model has a $G \times Osp(1|2)$ symmetry.

5. Superconformal Symmetry and the N=2B Model

5.1. Superconformal Algebra

The methods developed to investigate the superconformal properties of the N=1 supersymmetric sigma model can be extended to describe the superconformal properties of the N=2B model. We shall give a general treatment of the problem and then we shall specialize to find the conditions that are necessary for such models to be invariant under the subgroup $SU(1,1|1)$ of the superconformal group of the $N = 2B$ superspace. To describe the $su(1,1|1)$ algebra, we begin with the algebra $osp(1|2)$ in (4.2) and relabel the generator of the supersymmetry transformations and the generator of the special superconformal transformations by $Q_0 = Q$ and by

$S_0 = S$, respectively. Then we add second supersymmetry generator Q_1 and second special superconformal generator S_1 together with the R symmetry generator of the N=2B supersymmetry algebra. The non-vanishing (anti-)commutators of the $su(1,1|1)$ algebra are

$$
\begin{gathered}
\{Q_m, Q_n\} = 2H\delta_{mn}, \qquad [Q_m, L] = Q_m, \qquad \{S_m, Q_n\} = L\delta_{mn} + \epsilon_{mn}R, \\
\{S_m, S_n\} = 2T\delta_{mn}, \qquad [L, T] = T, \qquad [L, S_m] = 2S_m, \\
[Q_m, T] = \frac{1}{2}S_m, \qquad [S_m, H] = -\frac{1}{2}Q_m, \qquad [H, L] = H, \qquad [H, T] = \frac{1}{2}L, \\
[L, T] = T, \qquad [R, Q_m] = \epsilon_{mn}Q_n, \qquad [R, S_m] = \epsilon_{mn}S_n \ ,
\end{gathered}
\tag{5.1}
$$

where $m, n = 0, 1$.

5.2. Invariance of the Action

To construct superconformal transformations that obey the above algebra, one should combine the extended N=2B supersymmetry transformations of the one-dimensional sigma models [11] with the (2,0) supersymmetry transformations of the two-dimensional massive sigma models [23] and the superconformal transformations of the N=1 models of the previous section. These lead to the ansatz for the extended superconformal transformations

$$
\begin{aligned}
\delta_\eta q^i &= -D\eta c(t) I^i{}_j Dq^j + \eta Y^i(q,t) \\
\delta_\eta \psi^a &= -D\eta c(t) I^a{}_b \nabla\psi^b - (B_i)^a{}_b \delta_\eta q^i \psi^b + \eta L^a{}_b \psi^b + mD\eta G^a \ ,
\end{aligned}
\tag{5.2}
$$

where η is a commuting constant infinitesimal parameter such that

$$
\eta| = 0 \ , \tag{5.3}
$$

$c(t)$ is a function of $\mathbb{R}$, $I^i{}_j$ is an endomorphism of the tangent bundle of the sigma model manifold, $I^a{}_b$ and $L^a{}_b$ are endomorphisms* of the E vector bundle, and

* A special case is $I^a{}_b = 0$, see [25].

G^a is a section of E. We have chosen the same function c in the transformations of both q and ψ superfields in (5.2) because they are generated by the same odd vector field in superspace.

The conditions for the part of the action (4.13) that involves only the superfield q to be invariant under the transformations (5.2) are the following:

$$
\begin{aligned}
g_{ik}I^k{}_j + (j,i) &= 0 \\
\nabla^{(+)}_{(i} I_{j)k} &= 0 \\
\partial_{[i}\big(I^m{}_j c_{|m|kl}\big) - 2I^m{}_{[i}\partial_{[m} c_{jkl} &= 0 \\
\partial_t c I_{ij} + \nabla_{[i}Y_{j]} + Y^m c_{mij} + cI^m{}_{[i}F_{j]m} &= 0 \\
Y^j F_{ji} + \partial_t Y^j g_{ji} &= 0 \\
\nabla_{(i}Y_{j)} &= 0 \\
-\frac{1}{2}\partial_t c\nabla_{[i}I_{jk]} + \frac{1}{6}\partial_t c\nabla^{(+)}_{[i} I_{jk]} + \frac{1}{6}Y^m (dc)_{mijk} - c\nabla_{[i}\big(I^m{}_j F_{k]m}\big) &= 0 \ .
\end{aligned}
\tag{5.4}
$$

The first three conditions are required from extended N=2B supersymmetry and they have been extensively investigated in the literature. Part of fourth condition is also due to the requirement for invariance of the action (4.13) under extended supersymmetry and the rest is due to the requirement for invariance of (4.13) under extended superconformal transformations. Next the conditions for invariance of part of the action (4.13) that involves the superfield ψ under the transformations (5.2) are as follows:

$$
\begin{aligned}
h_{ac}I^c{}_b + (b,a) &= 0 \\
\nabla_i I^a{}_b &= 0 \\
I^m{}_i G_{mj}{}^a{}_b - (j,i) &= 0 \\
s_a G^a &= \text{const} \\
-c\nabla_j s_a I^j{}_i + c\nabla_i\big(s_b I^b{}_a\big) + \nabla_i G^b h_{ba} &= 0 \\
L^a{}_b &= 0 \\
Y^j G_{ji}{}^a{}_b &= 0 \\
Y^i \nabla_i s_a + \partial_t c s_b I^b{}_a &= 0 \ .
\end{aligned}
\tag{5.5}
$$

The first four conditions and part of the fifth have been expected and arise from the requirement of the invariance of the action (4.13) under N=2B extended supersymmetry, the rest are due to the additional requirement of superconformal invariance.

5.3. Closure of the Algebra

We shall first investigate the closure of the algebra of transformations on q. A direct computation reveals that

$$\begin{aligned}[\delta_\eta, \delta_\zeta] q^i = & -D\eta D\zeta c c' N(I)^i{}_{jk} Dq^j Dq^k \\ & +2D\eta D\zeta c c' (I^2)^i{}_j \partial_t q^i \\ & +D\eta D\zeta I^i{}_j (c' Y^j + c Y'^j) \\ & +D\eta D\zeta (c' Dc + c Dc')(I^2)^i{}_j Dq^j \\ & -\eta D\zeta c' \mathcal{L}_Y I^i{}_j Dq^j + \zeta D\eta c \mathcal{L}_{Y'} I^i{}_j Dq^j \ ,\end{aligned} \tag{5.6}$$

where $N(I)$ is the Nijenhuis tensor of the endomorphism I, and (c, Y) and (c', Y') are associated with the transformations with parameters η and ζ, respectively. The endomorphism I associated with the η transformation can be chosen to be different from a I' associated with ζ transformation. However, we have taken them to be equal $I = I'$ since this is a minimal choice.

The commutator of two transformations (5.2) on the superfield ψ is

$$\begin{aligned}[\delta_\eta, \delta_\zeta] \psi^a = & - 2D\eta D\zeta c c' (I^2)^a{}_b \nabla_t \psi^b - B_i{}^a{}_b [\delta_\eta, \delta_\zeta] q^i \psi^b \\ & + mD\eta D\zeta (c' Dc + c Dc')(I^2)^a{}_b s^b \\ & + m\eta D\zeta Y^i \nabla_i G'^a - mD\eta \zeta Y'^i \nabla_i G^a \\ & - mD\eta D\zeta (c' I^a{}_b \nabla_i G^b + c' I^a{}_b \nabla_i G'^b - c I^j{}_i \nabla_j G'^a - c' I^j{}_i \nabla_j G^a) Dq^i \\ & - D\eta D\zeta (c' Dc + c Dc')(I^2)^a{}_b \mathcal{S}^b \ ,\end{aligned} \tag{5.7}$$

where

$$\mathcal{S}^a = -\nabla \psi^a + h^{ab} s_b \tag{5.8}$$

is the field equation of the ψ superfield.

It remains to compute the commutator of the superconformal transformations of the N=1 model with the extended ones we have given above. We find that

$$\begin{aligned}[\delta_\eta, \delta_\zeta] q^i &= -\eta D\zeta c' \mathcal{L}_X I^i{}_j D q^j \\ &+ D\eta D\zeta c' D c I^i{}_j Q q^j + D\eta D\zeta c Q c' I^i{}_j D q^j \\ &+ D\eta D\zeta (c' I^i{}_j X^j + c Y'^i)\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}[\delta_\eta, \delta_\zeta] \psi^a &= D\eta D\zeta (c' D c + c D c') I^a{}_b \nabla \psi^b \\ &- B_i{}^a{}_b [\delta_\eta, \delta_\zeta] q^i \psi^b \\ &- m \eta D\zeta X^i \nabla_i G'^a \ ,\end{aligned} \tag{5.10}$$

where (c, X, η) are associated with the N=1 superconformal transformations and (c', Y', G'^a, ζ) are associated with the extended superconformal transformations.

5.4. $SU(1,1|1)$ Invariant Models

To find the conditions for a $N = 2B$ supersymmetric sigma model to admit $SU(1,1|1)$ superconformal invariance, we shall examine each transformation generated by $SU(1,1|1)$ to be a symmetry separately. The transformations generated by Q_0 and S_0 have already been investigated in section (4.3). The transformation generated by Q_1 is given in (5.2) for

$$c = 1, \qquad Y = 0, \qquad L = 0 \ . \tag{5.11}$$

This leads to the standard extended supersymmetry transformations of [11]. The

conditions for the invariance of the action are

$$
\begin{aligned}
g_{ik}I^k{}_j + (j,i) &= 0 \\
\nabla^{(+)}_{(i} I_{j)k} &= 0 \\
\partial_{[i}\big(I^m{}_j c_{|m|kl}\big) - 2I^m{}_{[i}\partial_{[m} c_{jkl} &= 0 \\
I^m{}_{[i}F_{j]m} &= 0 \\
h_{ac}I^c{}_b + (b,a) &= 0 \\
\nabla_i I^a{}_b &= 0 \\
I^m{}_i G_{mj}{}^a{}_b - (j,i) &= 0 \\
s_a G^a &= \mathrm{const} \\
-\nabla_j s_a I^j{}_i + \nabla_i\big(s_b I^b{}_a\big) + \nabla_i G^b h_{ba} &= 0 \ .
\end{aligned}
\tag{5.12}
$$

In addition, the algebra of two extended supersymmetries closes to translations provided that

$$
\begin{aligned}
N(I) &= 0 \\
I^i{}_j I^j{}_k &= -\delta^i{}_k \\
I^a{}_b I^b{}_c &= -\delta^a{}_c \\
I^a{}_b \nabla_i G^b - I^j{}_i \nabla_j G^a &= 0 \ .
\end{aligned}
\tag{5.13}
$$

Most of the conditions in (5.12) and (5.13) are well known and have been derived in the context of one-dimensional sigma models in [11, 30, 12]. In particular they imply that the target space is a hermitian manifold⋆ equipped with a holomorphic vector bundle E. The line bundle associated to the magnetic coupling is also holomorphic. The commutator of extended supersymmetry generated by Q_1 with that of N=1 supersymmetry generated by Q_0 vanishes without any further conditions. Next the commutator of extended supersymmetry transformation generated by Q_1

⋆ Under certain conditions, the integrability of the complex structure can be lifted; see [27,28, 29, 22].

with that of special superconformal transformation

$$\begin{aligned} \delta_\eta q^i &= -D\eta t Q q^i + \eta X^i(q) \\ \delta_\eta \psi^a &= -D\eta t Q \psi^a \end{aligned} \tag{5.14}$$

generated by S_0 is

$$\begin{aligned} [\delta_\eta, \delta_\zeta] q^i &= \alpha\beta I^i{}_j X^j \\ [\delta_\eta, \delta_\zeta] \lambda^i &= -\alpha\beta \big[\mathcal{L}_X I^i{}_j + I^i{}_j\big]\lambda^i + \alpha\beta \partial_k (I^i{}_j X^j) \lambda^k \\ [\delta_\eta, \delta_\zeta] \psi^a &= -B_i{}^a{}_b [\delta_\eta, \delta_\zeta] q^i \psi^b \\ [\delta_\eta, \delta_\zeta] F^a &= -\alpha\beta I^a{}_b F^b - \alpha\beta X^i \nabla_i G^a + \alpha\beta G_{ij}{}^a{}_b I^i{}_k X^k \lambda^j \psi^b - B_i{}^a{}_b [\delta_\eta, \delta_\zeta] q^i \psi^b \ , \end{aligned} \tag{5.15}$$

where $\alpha = D\eta|$ and $\beta = D\zeta|$. The above commutator should close to the transformation generated by the R symmetry of the $N = 2B$ superalgebra. This is the case provided that

$$\begin{aligned} \mathcal{L}_X I^i{}_j &= 0 \\ X^i \nabla_i G^a &= 0 \\ G_{ij}{}^a{}_b I^i{}_k X^k &= 0 \ . \end{aligned} \tag{5.16}$$

The latter condition is not independent condition but it is implied by the fact that $G_{ij}{}^a{}_b$ is (1,1) form with respect to I and that the contraction of X with $G_{ij}{}^a{}_b$ vanishes.

Next let us consider the case of the special superconformal transformation generated by S_1. For this we take

$$\begin{aligned} c &= t \\ Y(t, q) &= Y(q) \ . \end{aligned} \tag{5.17}$$

The additional conditions required for invariance of the action are as follows:

$$
\begin{aligned}
I_{ij} + \nabla_{[i}Y_{j]} + Y^m c_{mij} &= 0 \\
Y^j F_{ji} &= 0 \\
\nabla_{(i}Y_{j)} &= 0 \\
-\frac{1}{2}\nabla_{[i}I_{jk]} + \frac{1}{6}\nabla^{(+)}_{[i}I_{jk]} + \frac{1}{6}Y^m(dc)_{mijk} &= 0 \\
\nabla_i G^a &= 0 \\
Y^j G_{ji}{}^a{}_b &= 0 \\
Y^i \nabla_i s_a + s_b I^b{}_a &= 0 \ .
\end{aligned}
\tag{5.18}
$$

Observe that G^a is parallel. This leads to a simplification of the equations (5.12), (5.13) and (5.16). From the last

equation in (5.18) together with the hermiticity of the fibre metric h, we find that the scalar potential is *invariant* under the isometry Y.

Some of the equations in (5.12) and in (5.18) can be expressed in different ways. In fact they can be further simplified by choosing as $\nabla^{(+)}$ the unique metric connection of the hermitian geometry with antisymmetric torsion such that $\nabla^{(+)}I = 0$, i.e M is a KT manifold.

Now first us consider the commutator of two transformations generated by S_1. This commutator closes to a transformation generated by T provided that

$$
\begin{aligned}
X^i &= I^i{}_j Y^j \\
\mathcal{L}_Y I^i{}_j &= 0 \ .
\end{aligned}
\tag{5.19}
$$

The rest of the commutators are satisfied without further conditions. In the subclass of models without the fermionic multiplet ψ and without a coupling to a magnetic field the above conditions for $SU(1,1|1)$ invariance are related to those of [13].

To summarize, $SU(1,1|1)$ superconformal invariance of a sigma model with action (4.13) requires that the sigma model manifold M is hermitian and the vector bundle E is holomorphic with compatible fibre metric and complex structure. In addition E admits a parallel section (which can be chosen to vanish) and a holomorphic section s. Moreover M admits a holomorphic homothetic motion generated by X and a holomorphic isometry generated by Y which are related as in (5.19). The scalar potential scales under the homothetic motion with weight -2 and it is invariant under the isometry.

6. Superconformal Symmetry and the N=4B Model

6.1. Superconformal Algebra

The investigation of the superconformal symmetries of the $N = 4B$ supersymmetric sigma model can be done in a way similar to that of the $N = 2B$ supersymmetric sigma model in the previous sections. We shall begin with a general treatment of the problem and then we shall specialize to find the conditions that are necessary for such models to be invariant under the subgroup $D(2,1|\alpha)$ of the superconformal group of the $N = 4B$ superspace. The non-vanishing (anti)-commutators of the $D(2,1|\alpha)$ superconformal subalgebra are as follows:

$$
\begin{aligned}
&\{Q_m, Q_n\} = 2H\delta_{mn}, \qquad [Q_m, L] = Q_m, \\
&\{S_m, S_n\} = 2T\delta_{mn}, \qquad [L, T] = T, \qquad [L, S_m] = 2S_m, \\
&[Q_m, T] = \frac{1}{2}S_m, \qquad [S_m, H] = -\frac{1}{2}Q_m, \qquad [H, L] = H, \qquad [H, T] = \frac{1}{2}L, \\
&[L, T] = T, \qquad [R^r_\pm, Q_m] = (t^r_\pm)_{mn}Q_n, \qquad [R^r_\pm, S_m] = (t^r_\pm)_{mn}S_n \\
&\{S_m, Q_n\} = L\delta_{mn} + \frac{\alpha}{1+\alpha}(t^r_+)_{mn}R^r_+ + \frac{1}{1+\alpha}(t^r_-)_{mn}R^r_- \\
&[R^r_\pm, R^s_\pm] = \pm 2\epsilon^{rst}R^t_\pm \ ,
\end{aligned}
\tag{6.1}
$$

where $\{S_m; m = 0, \dots, 3\}$ are the generators of the special superconformal transformations, $\{Q_m; m = 0, \dots, 3\}$ are the supersymmetry generators, $\{R^r_\pm; r = 1, 2, 3\}$

are the generators of the $SO(4)$ R-symmetry of the $N=4B$ supersymmetry algebra and

$$(t^r_\pm)_{mn} = 2\delta^0{}_{[m}\delta^r{}_{n]} \pm \epsilon^{0r}{}_{mn} \tag{6.2}$$

is a basis of self-dual and anti-self-dual two forms in $\mathbb{R}^4$. It remains to investigate the conditions under which the action (4.13) admits such extended superconformal symmetry. For this, we consider the transformations

$$\begin{aligned}\delta_\eta q^i &= -D\eta^r c_r(t)(I_r)^i{}_j Dq^j + \eta^r Y^i_r(q,t)\\ \delta_\eta \psi^a &= -D\eta^r c_r(t)(I_r)^a{}_b \nabla\psi^b - (B_i)^a{}_b\delta_\eta q^i\psi^b + \eta^r(L_r)^a{}_b\psi^b + mD\eta^r G^a_r \ ,\end{aligned} \tag{6.3}$$

where $\{c_r; r=1,2,3\}$ are functions on $\mathbb{R}$, $\{\eta^r; r=1,2,3\}$ are a commuting constant infinitesimal parameters such that $\eta_r|=0$, $\{(I_r)^i{}_j; r=1,2,3\}$ are endomorphisms of the tangent bundle of the sigma model manifold, $\{(I_r)^a{}_b; r=1,2,3\}$ and $\{(L_r)^a{}_b; r=1,2,3\}$ are endomorphisms of the E vector bundle and $\{G^a_r; r=1,2,3\}$ are sections of E.

6.2. Invariance of the Action

The conditions for the part of the action (4.13) that involves only the superfield q to be invariant under the transformations (6.3) are the following:

$$\begin{aligned} g_{ik}(I_r)^k{}_j + (j,i) &= 0\\ \nabla^{(+)}_{(i}(I_r)_{j)k} &= 0\\ \partial_{[i}\big((I_r)^m{}_j c_{|m|kl}\big) - 2(I_r)^m{}_{[i}\partial_{[m}c_{jkl} &= 0\\ \partial_t c_r(I_r)_{ij} + \nabla_{[i}(Y_r)_{j]} + Y^m_r c_{mij} + c_r(I_r)^m{}_{[i}F_{j]m} &= 0\\ Y^j_r F_{ji} + \partial_t Y^j g_{ji} &= 0\\ \nabla_{(i}(Y_r)_{j)} &= 0\\ -\frac{1}{2}\partial_t e\nabla_{[i}(I_r)_{jk]} + \frac{1}{6}\partial_t e\nabla^{(+)}_{[i}(I_r)_{jk]} + \frac{1}{6}Y^m_r(dc)_{mijk} - e\nabla_{[i}\big((I_r)^m{}_j F_{k]m}\big) &= 0 \ . \end{aligned} \tag{6.4}$$

The first three conditions are required from extended supersymmetry and they have been extensively investigated in the literature. Part of fourth condition is

also due to the requirement for invariance of the action (4.13) under extended supersymmetry and the rest is due to the requirement for invariance of (4.13) under extended superconformal transformations. Next the conditions for invariance of part of the action (4.13) that involves the superfield ψ under the transformations (6.3) are as follows:

$$
\begin{aligned}
h_{ac}(I_r)^c{}_b + (b,a) &= 0 \\
\nabla_i (I_r)^a{}_b &= 0 \\
(I_r)^m{}_i G_{mj}{}^a{}_b - (j,i) &= 0 \\
s_a G^a_r &= \text{const} \\
-c_r \nabla_j s_a (I_r)^j{}_i + c_r \nabla_i \big(s_b (I_r)^b{}_a\big) + \nabla_i G^b_r h_{ba} &= 0 \\
L^a_{r\,b} &= 0 \\
Y^j_r G_{ji}{}^a{}_b &= 0 \\
Y^i_r \nabla_i s_a + \partial_t c_r s_b (I_r)^b{}_a &= 0 \ .
\end{aligned}
\tag{6.5}
$$

The first four conditions and part of the fifth have been expected and arise from the requirement of the invariance of the action (4.13) under extended supersymmetry, the rest are due to the additional requirement of superconformal invariance.

6.3. Closure of the Algebra

We shall first investigate the closure of the algebra of transformations on q. A direct computation reveals that

$$
\begin{aligned}
[\delta_\eta, \delta_\zeta] q^i = & -D\eta_r D\zeta_s c_r c_s N(I_r, I_s)^i{}_{jk} Dq^j Dq^k \\
& +2D\eta_r D\zeta_s c_r c_s (I_{(r} I_{s)})^i{}_j \partial_t q^i \\
& +D\eta_r D\zeta_s (c_s (I_s)^i{}_j Y^j_r + c_r (I_r)^j{}_i Y^j_s) \\
& +D\eta_r D\zeta_s (c_s Dc_r (I_s I_r)^i{}_j + c_r Dc_s (I_r I_s)^i{}_j) Dq^j \\
& -\eta_r D\zeta_s c_s \mathcal{L}_{Y_r} (I_s)^i{}_j Dq^j + \zeta_s D\eta_r c_r \mathcal{L}_{Y_s} (I_r)^i{}_j Dq^j \ ,
\end{aligned}
\tag{6.6}
$$

where $N(I_r, I_s)$ is the Nijenhuis tensor of the endomorphisms I_r and I_s.

The commutator of two transformations on the superfield ψ is

$$
\begin{aligned}
[\delta_\eta, \delta_\zeta]\psi^a =& - 2D\eta_r D\zeta_s c_r c_s (I_{(r} I_{s)})^a{}_b \nabla_t \psi^b - B_i{}^a{}_b [\delta_\eta, \delta_\zeta] q^i \psi^b \\
&+ mD\eta_r D\zeta_s c_s Dc_r (I_s I_r)^a{}_b s^b + mD\eta_r D\zeta_s c_r Dc_s (I_r I_s)^a{}_b s^b \\
&+ m\eta_r D\zeta_s Y_r^i \nabla_i G_s^a - mD\eta_r \zeta_s Y_s^i \nabla_i G_r^a \\
&- mD\eta_r D\zeta_s (c_s (I_s)^a{}_b \nabla_i G_r^b + c_r (I_r)^a{}_b \nabla_i G_s^b \\
&- c_r (I_r)^j{}_i \nabla_j G_s^a - c_s (I_s)^j{}_i \nabla_j G_r^a) Dq^i \\
&- D\eta_r D\zeta_s c_s Dc_r (I_s I_r)^a{}_b \mathcal{S}^b - D\eta_r D\zeta_s c_r Dc_s (I_r I_s)^a{}_b \mathcal{S}^b \ ,
\end{aligned}
\tag{6.7}
$$

where

$$
\mathcal{S}^a = -\nabla\psi^a + h^{ab} s_b \tag{6.8}
$$

is the field equation of the ψ superfield.

The commutator of (6.3) transformations with those of N=1 superconformal symmetries of section (4.1) is

$$
\begin{aligned}
[\delta_\eta, \delta_\zeta] q^i =& -\eta D\zeta c'_r \mathcal{L}_X I^i_{rj} Dq^j \\
&+ D\eta D\zeta_r c'_r Dc I^i_{rj} Qq^j + D\eta D\zeta_r c Q c'_r I^i_{rj} Dq^j \\
&+ D\eta D\zeta_r (c'_r I^i_{rj} X^j + c Y'^i_r)
\end{aligned}
\tag{6.9}
$$

and

$$
\begin{aligned}
[\delta_\eta, \delta_\zeta]\psi^a =& D\eta D\zeta_r (c'_r Dc + c Dc'_r) I^a_{r\,b} \nabla\psi^b \\
&- B_i{}^a{}_b [\delta_\eta, \delta_\zeta] q^i \psi^b \\
&- m\eta D\zeta_r X^i \nabla_i G'^a_r \ ,
\end{aligned}
\tag{6.10}
$$

where (c, X, η) are associated with the N=1 superconformal transformations and $(c', Y'_r, G'^a_r, \zeta)$ are associated with the (6.3) superconformal transformations.

6.4. $D(2,1|\alpha)$-invariant models

To investigate the conditions for the $N=4B$ supersymmetric sigma models to admit a $D(2,1|\alpha)$ superconformal symmetry, we shall examine the conditions required for each transformation generated by $D(2,1|\alpha)$ to be a symmetry, separately. The conditions required for the invariance of the sigma models under the transformations generated by Q_0 and S_0 have already been investigated in section (4.3). The extended supersymmetry transformations generated by Q_r are given by (6.3) for

$$c_r = 1, \qquad Y_r = 0, \qquad L_r = 0 \ . \tag{6.11}$$

The conditions for the invariance of the action are then

$$\begin{aligned}
g_{ik}(I_r)^k{}_j + (j,i) &= 0 \\
\nabla^{(+)}_{(i}(I_r)_{j)k} &= 0 \\
\partial_{[i}\big((I_r)^m{}_j c_{|m|kl}\big) - 2(I_r)^m{}_{[i}\partial_{[m}c_{jkl} &= 0 \\
(I_r)^m{}_{[i}F_{j]m} &= 0 \\
h_{ac}(I_r)^c{}_b + (b,a) &= 0 \\
\nabla_i(I_r)^a{}_b &= 0 \\
(I_r)^m{}_i G_{mj}{}^a{}_b - (j,i) &= 0 \\
s_a G^a_r &= \text{const} \\
-\nabla_j s_a (I_r)^j{}_i + \nabla_i\big(s_b(I_r)^b{}_a\big) + \nabla_i G^b_r h_{ba} &= 0 \ .
\end{aligned} \tag{6.12}$$

The conditions for the closure of the algebra of the extended supersymmetry transformations are that

$$\begin{aligned}
N(I_r, I_s) &= 0 \\
I_{(r}I_{s)} &= -\delta_{rs} \\
I_{(r}I_{s)} &= -\delta_{rs} \\
(I_r)^a{}_b\nabla_i G^b_r - (I_r)^j{}_i\nabla_j G^b_r + (s,r) &= 0 \ .
\end{aligned} \tag{6.13}$$

Most of the conditions (6.12) and (6.13) also appear in the case of $N=4B$ supersymmetric sigma models without scalar potential in [11, 12]. In what follows we

shall choose the complex structures to obey the somewhat stronger condition

$$I_r I_s = -\delta_{rs} + \epsilon_{rst} I_t \ . \tag{6.14}$$

It was shown in [13] that (6.14) and the first three conditions in (6.12) imply that

$$\nabla^{(+)}_i (I_r)^j{}_k = 0 \ . \tag{6.15}$$

So the sigma model manifold admits a weak HKT structure [31]. Moreover, the fourth and seventh conditions in (6.12) imply that both the line bundle associated with the magnetic field and the vector bundle E are holomorhic with respect to all three complex structures $\{I_r\}$.

For the extended special superconformal transformations generated by S_r, we choose in (6.3)

$$c_r = t \ , \qquad Y_r(q,t) = Y_r(q) \ . \tag{6.16}$$

The additional conditions for the invariance of the action under such transformations are as follows:

$$\begin{aligned} I_{rij} + \nabla_{[i}(Y_r)_{j]} + Y^m_r c_{mij} &= 0 \\ Y^j_r F_{ji} &= 0 \\ \nabla_{(i}(Y_r)_{j)} &= 0 \\ -\frac{1}{2}\nabla_{[i}(I_r)_{jk]} + \frac{1}{6} Y^m_r (dc)_{mijk} &= 0 \\ \nabla_i G^a_r &= 0 \\ Y^j_r G_{ji}{}^a{}_b &= 0 \\ Y^i_r \nabla_i s_a + s_b (I_r)^b{}_a &= 0 \ . \end{aligned} \tag{6.17}$$

Observe that the above conditions imply that G^a_r are parallel. Using this, one can simplify some of the conditions in (6.12) and (6.13). In what follows we shall use that G^a_r are parallel. Some of the conditions in (6.17) can be expressed in a different way, e.g. the fourth condition using (6.15).

The commutator of two extended special superconformal transformations generated by S_r closes to special conformal transformations provided that

$$
\begin{aligned}
X^i &= (I_r)^i{}_j Y_r^j \\
\mathcal{L}_{Y_r}(I_s)^i{}_j + (s,r) &= 0 \ .
\end{aligned}
\tag{6.18}
$$

To find the transformations associated with the $R\pm$ generators of $D(2,1|\alpha)$, we shall next compute the commutator of the special superconformal transformations generated by S_n and the supersymmetry transformations generated by Q_m. Let us begin with the commutator of extended supersymmetry transformations generated by Q_s with the extended special superconformal transformations generated by S_r. We find that

$$
\begin{aligned}
[\delta_\alpha,\delta_\beta]q^i =& -2\alpha_r\beta_s\delta_{rs}t\partial_t q^i + \alpha_r\beta_s\delta_{rs}X^i - \alpha_r\beta_s\epsilon_{rst}(I_t)^i{}_j X^j) \\
[\delta_\alpha,\delta_\beta]\lambda^i =& -2\alpha_r\beta_s\delta_{rs}t\partial_t\lambda^i + \alpha_r\beta_s\delta_{rs} - \epsilon_{rst}(I_t)^i{}_j\lambda^j \\
&+ \alpha_r\beta_s\delta_{rs}\partial_j X^i\lambda^j - \epsilon_{rst}\partial_k\big((I_t)^i{}_j)X^j\big)\lambda^k \\
&- \alpha_r\beta_s\mathcal{L}_{Y_s}(I_r)^i{}_j\lambda^j \\
[\delta_\alpha,\delta_\beta]\psi^a =& 2\alpha_r\beta_s\delta_{rs}t\nabla_t\psi^a - B_i{}^a{}_b[\delta_\alpha,\delta_\beta]q^i\psi^b \\
[\delta_\alpha,\delta_\beta]F^a =& 2\alpha_r\beta_s\delta_{rs}t\nabla_t F^a - 2\alpha_r\beta_s(I_rI_s)^a{}_bF^b - B_i{}^a{}_b[\delta_\alpha,\delta_\beta]q^iF^b \ ,
\end{aligned}
\tag{6.19}
$$

where $\alpha_r = D\eta_r|$ and $\beta_s = D\zeta_s|$ are the infinitesimal parameters of the supersymmetry and special superconformal transformations, respectively. The commutator of the first supersymmetry transformation generated by Q_0 with the extended superconformal ones generated by S_r is

$$
\begin{aligned}
[\delta_\alpha,\delta_\beta]q^i =& -\alpha\beta_s(I_s)^i{}_jX^j \\
[\delta_\alpha,\delta_\beta]\lambda^i =& \alpha\beta_s(I_s)^i{}_j\lambda^j \\
&- \alpha\beta_s\partial_k\big((I_s)^i{}_jX^j\big)\lambda^k \\
[\delta_\alpha,\delta_\beta]\psi^a =& -B_i{}^a{}_b[\delta_\alpha,\delta_\beta]q^i\psi^b \\
[\delta_\alpha,\delta_\beta]F^a =& \alpha\beta_s(I_s)^a{}_bF^b - B_i{}^a{}_b[\delta_\alpha,\delta_\beta]q^iF^b \ ,
\end{aligned}
\tag{6.20}
$$

The commutator of the special superconformal transformation generated by S_0

with extended supersymmetry ones generated by Q_r is

$$
\begin{aligned}
[\delta_\alpha, \delta_\beta] q^i =& \alpha\beta_r (I_r)^i{}_j X^j \\
[\delta_\alpha, \delta_\beta] \lambda^i =& -\alpha\beta_r \mathcal{L}_X (I_r)^i{}_j \lambda^j - \alpha\beta_r (I_r)^i{}_j \lambda^j \\
& +\alpha\beta_r \partial_k \big((I_r)^i{}_j X^j \big) \lambda^k \\
[\delta_\alpha, \delta_\beta] \psi^a =& -B_i{}^a{}_b [\delta_\alpha, \delta_\beta] q^i \psi^b \\
[\delta_\alpha, \delta_\beta] F^a =& -\alpha\beta_r (I_r)^a{}_b F^b - B_i{}^a{}_b [\delta_\alpha, \delta_\beta] q^i F^b \ .
\end{aligned}
\tag{6.21}
$$

In order the part of the anti-commutator $\{S_m, Q_n\}$ involving the R generators to be antisymmetric, we take

$$
\mathcal{L}_X I_r = 0 \ . \tag{6.22}
$$

Next following closely the analysis in [13], we choose

$$
\mathcal{L}_{Y_s} I_r = h\epsilon_{rst} I_t \ , \tag{6.23}
$$

where h is a real number. Then the part of the commutators on λ involving the R-symmetries can be written schematically as

$$
[\delta_\alpha, \delta_\beta] \lambda^i \equiv [\alpha^m Q_m, \beta^n S_n] \lambda^i = \alpha_m \beta_n \delta_{mn} L \lambda^i + \alpha_m \beta_n t^r_{mn} R_r \lambda^i \ , \tag{6.24}
$$

where

$$
\begin{aligned}
t^r_{0s} &= \delta^r{}_s \\
t^r_{s0} &= -\delta^r{}_s \\
t^r_{st} &= -(1+h)\epsilon_{rst} \ .
\end{aligned}
\tag{6.25}
$$

Rewriting t^r in terms of the self-dual (t^r_+) and anti-self-dual (t^r_-) basis of two-forms, we find that

$$
t^r = -\frac{h}{2}(t^r_+) + \frac{2+h}{2}(t^r_-) \ . \tag{6.26}
$$

Comparing this with the superconformal algebra $D(2,1|\alpha)$ in (6.1), we find that

$$
\alpha = -\frac{h}{2+h} \tag{6.27}
$$

Finally we observe that q, ψ and F transform only under the R_+ generators of

$D(2,1|\alpha)$ while λ transforms under both R_+ and R_- generators of $D(2,1|\alpha)$. The rest of the commutators are satisfied without further conditions. In the case of models without fermionic superfields ψ and a coupling to a magnetic field A, the above conditions for $D(2,1|\alpha)$ are related to those derived in [13].

To summarize, $D(2,1|\alpha)$ superconformal invariance of a sigma model with action (4.13) requires that the sigma model manifold M is HKT and the vector bundle E is tri-holomorphic with compatible fibre metric and hypercomplex structure. In addition E admits three parallel sections (which can be chosen to vanish) and a tri-holomorphic section s. Moreover M admits a holomorphic homothetic motion generated by X and three holomorphic isometries generated by Y_r which are related as in (6.18). The scalar potential scales under the homothetic motion with weight -2 and it is invariant under the three isometries.

7. Conclusions

We have given the conditions for a spinning particle coupled to a magnetic field and with a scalar potential to be invariant under conformal and superconformal symmetries. In particular, we have presented the conditions for such models to admit $SL(2,\mathbb{R})$, $Osp(1|2)$, $SU(1,1|1)$ or $D(2,1|\alpha)$ (super)conformal symmetries. We have given several examples of such systems that include a particle propagating in a Dirac monopole background and a particle propagating in a conical spacetime. Moreover we have found an interpretation of the conformal transformations using the geometry of bundles.

In the superconformal case, we considered a special class of one-dimensional sigma model Lagrangians with a scalar potential. There are many other possibilities to explore. It is already known that some of these cases have applications in the context of the moduli spaces of four-dimensional supersymmetric black holes [15]. Therefore it would be of interest to explore the superconformal properties of these more general models described by the Lagrangian of appendix B. We leave this work for the future.

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APPENDIX A

The geometry of conformal transformations

There is a geometric interpretation of the conformal transformations we are considering in the bosonic spinning particle. To see this, we consider $P = \mathbb{R} \times M$ as a bundle over $\mathbb{R}$ with respect to the obvious projection $\pi_{\mathbb{R}}$, where $\mathbb{R}$ is the worldline and M is the sigma model target space. Let σ be a section of P, then the sigma model maps q can be written as

$$q = \pi_{M} \sigma \tag{A.1}$$

where π_M is the obvious projection of P onto M. Next consider the bundle isomorphisms α of P,

$$\alpha(t, x) = \left(f(t), g(t, x)\right) , \tag{A.2}$$

where $t \in \mathbb{R}$, $x \in M$, f is a diffeomorphism of $\mathbb{R}$ and g is a diffeomorphism of M for every t. The bundle isomorphism α induces a transformation U_α on the space of sections of P as follows:

$$(U_\alpha \sigma)(x) = \alpha\big(\sigma(f^{-1}(x))\big) . \tag{A.3}$$

Observe that U_α takes a section σ at the point x to another section $U_\alpha \sigma$ at the same point, i.e. it shifts the sections along the fibre M. Composing U_α with the projection on M, we induce a transformation on the sigma model maps which we

also denote by U_α, i.e.

$$(U_\alpha(q))(x) = \pi_{_M}(U_\alpha \sigma)(x) \ . \tag{A.4}$$

Next consider an one-parameter family α_s of isomorphisms such that

$$\alpha_{s=0} = Id_P \ . \tag{A.5}$$

The vector fields on P that generate such one-parameter families of isomorphisms are

$$A = a(t)\partial_t + X^i(t,x)\partial_i \ . \tag{A.6}$$

Moreover a direct computation reveals that the vector fields on P which are generated by the associated transformations U_{α_s} on the sections of P are

$$\tilde{A} = -a(t)\partial_t \big(\pi_{_\mathbb{R}} \sigma\big)^i \partial_i + X^i(t, \pi_{_\mathbb{R}} \sigma)\partial_i \tag{A.7}$$

or on the sigma model maps q

$$\tilde{A} = -a(t)\partial_t q^i \partial_i + X^i(t,q)\partial_i \ . \tag{A.8}$$

Clearly $\tilde{A}$ generates the infinitesimal transformations that we have investigated in section two on the sigma model maps q.

APPENDIX B

The $N = 1$ Supersymmetric Action

The most general $N = 1$ spinning particle action coupled to a magnetic filed and with the scalar potential is

$$
\begin{aligned}
S = -\int dt d\theta \big[& \frac{1}{2} g_{ij} Dq^i \partial_t q^j + \frac{1}{3!} c_{ijk} Dq^i Dq^j Dq^k - \frac{1}{2} h_{ab} \psi^a \nabla \psi^b \\
& + \frac{1}{2} m_{iab} Dq^i \psi^a \psi^b + \frac{1}{2} n_{ija} Dq^i Dq^j \psi^a + \frac{1}{3!} l_{abc} \psi^a \psi^b \psi^c \\
& + f_{ia} \partial_t \psi^a + A_i Dq^i + m s_a \psi^a \big] \ ,
\end{aligned} \tag{B.1}
$$

where q is a bosonic and ψ is a fermionic $N = 1$ superfield, respectively. This action apart from the last two terms has been proposed in [11]. The term involving A is a standard supersymmetric extension of the coupling of a charged particle to a magnetic field. The last term involves the addition of a scalar potential in the action. Such a term has been first proposed in the context of two-dimensional sigma models in [23] and it can be easily adapted to this case. It is worth pointing out that the addition of the scalar potential necessarily involves the superfield ψ.

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